REDUCTION OF A CLASS OF OPTIMUM CONTROL PROBLEMS TO THE SIMPLEST VARIATIONAL PROBLEM

(PRIVEDENIE ODNOGO KLASSA ZADACH OPTIMAL'NOGO REGULIROVANIIA K PROSTEISHEI VARIATSIONNOI ZADACHE)

PMM Vol.28, № 2, 1964, pp.375-380

G.N.MIL'SHTEIN
(Sverdlovsk)

(Received November 15, 1963)

1. Statement of the problem. We consider the problem of minimizing the functional (see, for example, [1])

$$J = \int_0^{t_1} f_0(x)\,dt \qquad (1.1)$$

Here $x = (x_1(t),\ldots,x_n(t))$ is an n-dimensional vector whose variation with respect to time is governed by a system of differential equations expressed in vector form

$$dx / dt = f(x, u) \qquad (1.2)$$

Here the control function u is an r-dimensional vector whose instantaneous values belong to some set U in r-dimensional Euclidean space. The set U is given by means of the inequality $\rho(u_1,\ldots,u_r) \leqslant m$, where $\rho(u_1, \ldots, u_r)$ is a continuously differentiable function. For the sake of definiteness, we shall consider the function $f_0(x)$ in (1.1) to be positive everywhere except at $x = 0$.

The control function $u(t)$ must be chosen so that the trajectory of the system (1.2), beginning at the point x_0 at zero time, passes at some instant $t_1 > 0$ through a prescribed point x_1 and so that at the same time the functional (1.1) takes on its minimum value over all such control functions u.

Most optimum-control problems are so formulated that the second point is usually fixed, while the first takes on an arbitrary position. In what follows the first point (for the sake of definiteness, we shall consider this the origin) is fixed and the second is chose arbitrarily. To reduce the usual problem to the one considered above without changing the time variation, it is sufficient to replace the system (1.2) and the functional (1.1), respectively, by

$$dx / dt = -f(x, u), \qquad J = \int_{-t_1}^{0} f_0(x)\,dt \qquad (1.3)$$

The control function $u(t)$ and the trajectory $x(t)$ of problem (1.1) – (1.2) correspond in this case to the control function $u(-t)$ and the trajectory $x(-t)$ of problem (1.3), and vice versa. The Bellman equation [2] for problem (1.1) – (1.2) is of the form

$$\min_u (p, f(x, u)) + f_0(x) = 0, \qquad (u \subset U, p = \partial J / \partial x) \qquad (1.4)$$

For problem (1.3) this can be represented as

$$\max_u (p, -f(x, u)) - f_0(x) = 0 \qquad (u \subset U) \tag{1.5}$$

In (1.4) and (1.5) the function $J(x_1, \ldots, x_n)$ is the Bellman function for the optimum problem and

$$J(x_1, \ldots, x_n) = \min_u \int_0^{t_1} f_0(x)\, dt \tag{1.6}$$

The vector $p = \partial J / \partial x$ has the coordinates $p_1 = \partial J / \partial x_1 \ldots, p_n = \partial J / \partial x_n$, and Expression $(p, -f(x,u))$ is the scalar product of the vector p and $-f(x,u)$.

Let the problem under consideration be such that Equation (1.5), after elimination of the control function u from the maximization condition, reduces to the form

$$L(x, p) = H(x, p) - f_0(x) = -(p, f(x, u(x, p))) - f_0(x) = 0 \tag{1.7}$$

and the functions $u(x, p)$ and $H(x, p)$ are sufficiently smooth functions in some region S of variation of the variables x and p. Hereafter we shall use the term "Bellman equation" for Equation (1.7), just as for Equation (1.5).

It is readily shown that the function $H(x, p)$ must be a positive homogeneous function of first degree in the variables p. We know that

$$\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} = -\sum_{i=1}^{n} p_i \frac{\partial (p, f(x, u(x, p)))}{\partial p_i} = -\sum_{i=1}^{n} p_i f_i(x, u(x, p)) - \\ - \sum_{i=1}^{r} p_i \sum_{k=1}^{r} \frac{\partial (p, f(x, u(x, p)))}{\partial u_k} \frac{\partial u_k}{\partial p_i} \tag{1.8}$$

But the sum

$$\sum_{k=1}^{r} \frac{\partial (p, f(x, u(x, p)))}{\partial u_k} \frac{\partial u_k}{\partial p_i} \qquad (i = 1, \ldots, n)$$

vanishes, since (1.5) is maximized, and (1.8) therefore becomes the well-known Euler formula. For the first-order partial differential Equation (1.7) we set up the characteristic system of equations [3]. It will be of the form

$$\frac{dx_i}{dt} = \frac{\partial L}{\partial p_i} = -f_i(x, u(x, p)) \qquad \frac{dp_i}{dt} = -\frac{\partial L}{\partial x_i} = \sum_{k=1}^{n} p_k \frac{\partial f_k(x, u(x, p))}{\partial x_i} + \frac{\partial f_0}{\partial x_i} \tag{1.9}$$

Condition (1.5) and the system of equations (1.9) may also be obtained from the maximum principle by Pontriagin's method [1]..

We shall consider the functions $F(x,x^\cdot)$ which have the following properties [4]: (a) the region of definition of the function $F(x,x^\cdot)$ is a region R in the $2n$-dimensional space of the variables x and $x^\cdot$, such that for each point $(x,x^\cdot)$ it also contains every point of the form $(x, \sigma x^\cdot)$, where $\sigma > 0$; (b) the function $F(x,x^\cdot)$ is sufficiently smooth; (c) the function $F(x,x^\cdot)$ must be a positive homogeneous function of first degree in the variables $x^\cdot$, that is to say, Equation $F(x, \sigma x^\cdot) = \sigma F(x,x^\cdot)$ holds for every point $(x,x^\cdot)$ in R and for all $\sigma > 0$.

Let us now try to find a function $F(x,x^\cdot)$ such that, besides the above properties, it satisfies the following condition. Using the function $F(x,x^\cdot)$, we set up the functional

$$J = \int_C F(x, x^\cdot)\, dt \tag{1.10}$$

and consider the problem of minimizing this functional over all curves C passing through the origin and through some point x, i.e. the simplest variational problem in parametric form. We shall require that the extremals of the problem (1.10) coincide with the extremal of the optimum problem (1.3). A sufficient condition for this is that the Euler equations for problem (1.10), when written in canonical form, coincide with Equations (1.9) and that the additional condition imposed on the choice of the parameter of the variational problem (1.10), after the introduction of the canonical variables, assume the form (1.7). Thus, after the function $F(x, x^{\cdot})$ has been found, the optimum control problem (1.3) reduces to the simplest problem of the calculus of variations in parametric form.

2. In [4] a method is shown for finding the Hamilton function and constructing a system of canonical equations for the variational problem in parametric form (1.10). By slightly modifying and expanding the reasoning used in [4], we can find a method for determining the function $F(x, x^{\cdot})$ if we know the form of the function $L(x, p)$ from (1.7). Considering problem (1.10), we introduce the function

$$G(x, x^{\cdot}, l) = F(x, x^{\cdot}) + l\left(\frac{F(x, x^{\cdot})}{f_0(x)} - 1\right) \tag{2.1}$$

Let us consider the Legendre transformation [5] using the variables $x_1^{\cdot}, \ldots, x_n^{\cdot}, l$ for the function $G(x, x^{\cdot}, l)$

$$p_i = G'_{x_i^{\cdot}} = F'_{x_i^{\cdot}}\left(1 + \frac{l}{f_0(x)}\right), \qquad \lambda = G_l' = \frac{F}{f_0(x)} - 1 \tag{2.2}$$

If the Jacobian of the transformation (2.2)

$$d = \frac{(f_0(x) + l)^{n-1}}{[f_0(x)]^{n+1}} \begin{vmatrix} F''_{x_1^{\cdot}x_1^{\cdot}} & \cdots & F'_{x_1^{\cdot}x_n^{\cdot}} & F'_{x_1^{\cdot}} \\ \cdots & \cdots & \cdots & \cdots \\ F''_{x_n^{\cdot}x_1^{\cdot}} & \cdots & F''_{x_n^{\cdot}x_n^{\cdot}} & F'_{x_n^{\cdot}} \\ F'_{x_1^{\cdot}} & \cdots & F'_{x_n^{\cdot}} & 0 \end{vmatrix} \tag{2.3}$$

is non-zero in some region R of the variables $x, x^{\cdot}$ and for $|l| < 1$, then Formulas (2.2) can be inverted.

It should be noted that the requirement that the determinant d be non-zero imposes an additional condition on the function $F(x, x^{\cdot})$. Thus, we know that the class of functions $F(x, x^{\cdot})$ includes no functions for which the determinant d vanishes, for example, linear functions in the variables $x_1^{\cdot}, \ldots, x_n^{\cdot}$.

Let the following condition hold from (2.2):

$$x_i^{\cdot} = Q_i(x, p, \lambda), \qquad l = L(x, p, \lambda) \tag{2.4}$$

The function $\Phi(x,p,\lambda)$, the dual of the function $G(x, x^{\cdot}, l)$ under the Legendre transformation, has the form

$$\Phi(x, p, \lambda) = \left[\sum_{i=1}^{n} x_i^{\cdot} p_i + l\lambda - G\right]^{x_i^{\cdot}=Q_i,\ l=L} =$$

$$= \left[\sum_{i=1}^{n} x_i^{\cdot} G'_{x_i^{\cdot}} + l\left(\frac{F}{f_0} - 1\right) - G\right]^{x_i^{\cdot}=Q_i,\ l=L} = L(\lambda + 1) \tag{2.5}$$

In deriving (2.5) we used the property of homogeneity of the function $F(x, x^{\cdot})$. Since the Legendre transformation is involutory, it follows that by applying this transformation to the function $\Phi(x, p, \lambda)$ we arrive at the variables $x^{\cdot}, l$ and the function $G(x, x^{\cdot}, l)$. The variables $x^{\cdot}$ will be related to p and λ, if we use the function $\Phi(x, p, \lambda)$, as follows:

$$x_i^{\cdot} = \frac{\partial\Phi}{\partial p_i} = \frac{\partial L}{\partial p_i}(\lambda + 1), \qquad l = \frac{\partial\Phi}{\partial\lambda} = L + \frac{\partial L}{\partial\lambda}(\lambda + 1) \tag{2.6}$$

In order to find the function $G(x, x^{\cdot}, l)$ we must invert Formulas (2.6). A sufficient condition for this is that the Jacobian Δ of the transformation (2.6) which, as will be explained later, is equal to the determinant

$$\Delta = (\lambda + 1)^{n-1} \begin{vmatrix} L''_{p_1p_1} & \ldots & L''_{p_1p_n} & L'_{p_1} \\ \ldots & \ldots & \ldots & \ldots \\ L''_{p_np_1} & \ldots & L''_{p_np_n} & L'_{p_n} \\ L'_{p_1} & \ldots & L'_{p_n} & 0 \end{vmatrix} \tag{2.7}$$

be non-zero in some region S of variation of variables x, p and for $|\lambda| < 1$. It should be noted that the requirement that Δ be non-zero is an important additional condition which restricts the class of functions $L(x, p)$ under consideration, and hence the class of problems under consideration as well. Comparing the last equations of (2.4) and (2.6), we can convince ourselves that $\partial L / \partial \lambda = 0$ for $|\lambda| < 1$, so that L is independent of λ. To find the other properties of the function $L(x, p)$, we shall consider Formula

$$\sum_{i=1}^{n} p_i \frac{\partial \Phi}{\partial p_i} = \sum_{i=1}^{n} p_i \frac{\partial L}{\partial p_i} (\lambda + 1) \tag{2.8}$$

On the other hand,

$$\sum_{i=1}^{n} p_i \frac{\partial \Phi}{\partial p_i} = \Big[\sum_{i=1}^{n} p_i x_i^{\cdot} \Big]^{x_i^{\cdot} = Q_i} = \Big[\sum_{i=1}^{n} x_i^{\cdot} F_{x_i^{\cdot}}' \left(1 + \frac{l}{f_0}\right) \Big]^{x_i^{\cdot} = Q_i,\ l = L}$$

Finally, using the property of homogeneity of the function $F(x, x^{\cdot})$ and the last equation in (2.2), we find, that

$$\sum_{i=1}^{n} p_i \frac{\partial \Phi}{\partial p_i} = (L + f_0)(\lambda + 1) \tag{2.9}$$

From Equations (2.8) and (2.9) it follows that

$$\sum_{i=1}^{n} p_i \frac{\partial L}{\partial p_i} = L + f_0 \tag{2.10}$$

and this equation indicates that $L + f_0(x)$ is a positive homogeneous function of the first degree in the variables p. Conversely, if $L(x, p)$ has the above properties, i.e. if the function $L + f_0$ is a positive homogeneous function of the variables p and L is independent of λ, then by using L to construct the function $\Phi(x, p, \lambda) = L(x, p)(\lambda + 1)$ and applying to $\Phi(x, p, \lambda)$ the Legendre transformation with respect to the variables p and λ, we obtain the function $G(x, x^{\cdot}, l)$. The function $F(x, x^{\cdot})$, found by means of $G(x, x^{\cdot}, l)$ using Formula

$$F(x, x^{\cdot}) = \frac{f_0 (G + l)}{f_0 + l} \tag{2.11}$$

will be a positive homogeneous function of the first degree in the variables $x^{\cdot}$ and will be independent of l. This statement may be derived by a reasoning similar to that applied in deriving the properties of the function L. In this process we also find

$$F(x, x^{\cdot}) = f_0(x)(1 + \Lambda(x, x^{\cdot})) \tag{2.12}$$

where $\Lambda(x, x^{\cdot})$ is the variable λ found from (2.6) and independent of l.

If we assume, furthermore, that the function $L(x, p)$, which is the left-hand side of the Bellman Equation (1.7), has a non-zero determinant

$$\Delta(\lambda + 1)^{1-n}$$

it will satisfy all of the above conditions. As has been explained , if we know $L(x, p)$, we can find the integrand $F(x, x^\cdot)$, of some parametric-form variational problem (1.10). Selecting the parameter t along the extremals of problem (1.10) so as to satisfy the equality $F(x, x^\cdot) = f_0(x)$, we make the parameter λ in the transformations (2.2) vanish. Set $I = 0$, L also vanishes. Repeating exactly the same reasoning as is found in [4], p. 160, we can convince ourselves that to each extremal of problem (1.10) for which the parameter t is so chosen that $F(x, x^\cdot) = f_0(x)$ along the extremal, there corresponds a solution $x(t)$, $p(t)$ of the characteristic system of equations (1.9) along which Equation (1.7) is satisfied. The converse is also true. It should be noted that the condition of non-zero determinants d and Δ may be modified somewhat.

First of all, by virtue of Equation $d \cdot \Delta = 1$ (see [5]), it is sufficient to require one of these two Jacobians to be non-zero. Secondly, this condition along the extremals is equivalent to the condition that the matrix

$$\|F_{x_i^\cdot}{}''_{x_j^\cdot}\| \quad (\text{correspondingly } \|L_{p_i}{}''_{p_j}\| = \|H_{p_i}{}''_{p_j}\|)$$

be of rank $n - 1$. This may be shown just as in Theorem 43.1 of [4]. To prove this, in the first case, we must use the condition that the function

$$F = x_1^\cdot F'_{x_1^\cdot} + \ldots + x_n^\cdot F'_{x_n^\cdot} = f_0\ (x)$$

does not vanish along the extremals, and in the second case, the condition that the function

$$H\ (x,\ p) = p_1\, \partial H\ /\ \partial p_1 + \ldots + p_n \partial H\ /\ \partial p_n$$

does not vanish; by virtue of Equation (1.7), which is satisfied along the extremals, this function is also non-zero. Equation (1.7), which is the Bellman equation of the optimum problem (1.3), will at the same time be the Hamilton-Jacobi equation of the problem (1.10). Thus, the function J , which is the geodesic of the problem (1.10), coincides with Bellman function of the problem (1.3).

3. Let us consider briefly what result can be gained in the study of individual optimum control problems by this reduction to the simplest problem of the calculus of variation in parametric form. For the approximate calculation of optimum trajectories and of the Bellman function, a knowledge of the function $F(x, x^\cdot)$ enables us to apply a number of direct methods of calculus of variations. For the approximate determination of curves minimizing the functional(1.10), we evidently need not confine ourselves to the parametrization represented by condition

$$F\ (x,\ x^\cdot) = f_0\ (x) \tag{3.1}$$

since the functional (1.10) is independent of the choice of parametrization. The extremals which have been determined approximately will approximate the optimum trajectories in the phase space X , but in general they cannot be considered approximate optimum trajectories, since when a direct method is used, the selected parameter will not, as a rule, coincide with time t . However, we can always introduce into the approximate curves a parameter satisfying condition (3.1). For example, suppose that when direct methods are used the parameter selected is the arc length s and the approximate minimal is found in the form $x_n\ (s)$, $0 \leqslant s \leqslant s_1$. We shall find the function $s(t)$ defining the parameter s as a function of time t . The function $s(t)$ must satisfy Equation

$$F\ (x_n\ (s\ (t)),\ x_{n_t}'\ (s\ (t))) = f_0\ (x_n\ (s\ (t))) \tag{3.2}$$

or (since F is homogeneous in the derivatives $x^\cdot$) Equation

$$F\ (x_n\ (s),\ x_n'\ (s)) = f_0\ (x_n\ (s))\ t_s' \tag{3.3}$$

Noting that $t(s_1)$ must be zero, we find the function $t(s)$ from (3.3)

$$t(s) = \int_{s}^{s_1} \frac{F(x_n(s), x_n'(s))}{f_0(x_n(s))} ds \tag{3.4}$$

The desired function $s(t)$ is found as the inverse of the function $t(s)$. Thus, the approximate optimum trajectories $x_n(t) = x_n(s(t))$ are now known. This, in general, enables us to find the approximate optimum control functions. The Bellman function, which has been found approximately in some region of the phase space X as

$$\min_c \int_c F(x, x^{\cdot})\, dt \tag{3.5}$$

makes possible an approximate synthesis of the optimum control problem in this region. Conversion of the optimum control problem into the simplest variational problem can be very useful. In particular, it enables us to use the sufficient conditions of problem (1.10) in establishing the optimum capacity of the extremals. The existence of specific differential properties of the function $F(x, x^{\cdot})$ indicates definite differential properties in the extremals [4] of the function $J(x_1, \ldots, x_n)$, and hence the optimum trajectories and control functions.

We should also point out the importance of the fact that the function $F(x, x^{\cdot})$ is a positive definite function for any $x^{\cdot}$ in some region D of the phase space X including the origin. In this case the function $J(x, \ldots, x_n)$, equal to (3.5), will be positive definite in the region D and will satisfy the Bellman equation (1.7). Equation (1.7) indicates that the function J is the Liapunov function for the original control system considered (1.2) if, instead of the control functions u, we substitute the function $u(x, dJ/dx)$ from (1.7) into the right-hand part of the system (1.2). It follows from this that every state of the region D is controllable [6], and an estimate from below can be obtained for the controllability region of problem (1.1) – (1.2).

4. As an example, we cite a problem studied by Krasovskii [7]. In this problem the system of equations (1.2) is of the form

$$dx / dt = Ax + Bu \tag{4.1}$$

where A and B are n-dimensional matrices with constant coefficients, B being a nonsingular metrix and u an n-dimensional control vector. The control region U is a unit sphere, i.e. the control function $u = (u, \ldots, u_n)$ satisfies at any instant of time condition

$$(u, u) = u_1^2 + \ldots + u_n^2 \leqslant 1 \tag{4.2}$$

Krasovskii considered the problem from the viewpoint of speed of action, i.e. in (1.1) the function $f_0(x) \equiv 1$. All subsequent calculations are carried out without change if, instead of (4.1), we consider the system

$$dx / dt = f(x) + B(x)\, u \tag{4.3}$$

where $B(x)$ is an n-dimensional nonsingular matrix whose coefficients depend on the phase coordinates. Equation (1.5) for problem (1.1) – (4.1) is of the form

$$\max_u (p, -Ax - Bu) - f_0(x) = 0 \qquad ((u, u) \leqslant 1) \tag{4.4}$$

From the maximization condition (4.4) we can find the control function u as a vector function of the variables x and p and obtain a first-order partial differential equation independent of u, as was done in [7]; we obtain

$$u = -\frac{B^* p}{\sqrt{(BB^* p, p)}} \tag{4.5}$$

Substituting this control function into (4.4) and designating the matix BB^* as Γ, we find an equation of the type (1.7)

$$L(x, p) = -(p, Ax) + \sqrt{(\Gamma, p, p)} - f_0(x) = 0 \quad (4.6)$$

For the problem under consideration Equations (2.6) become

$$x^{\cdot} = \left(-Ax + \frac{\Gamma p}{\sqrt{(\Gamma p, p)}}\right)(\lambda + 1), \qquad l = -(p, Ax) + \sqrt{(\Gamma p, p)} - f_0(x) \quad (4.7)$$

In order to find the function $F(x, x^{\cdot})$ it is sufficient to express $\lambda + 1$ in terms of the variables x and $x^{\cdot}$ from (4.7). Rewriting the first n-equations from (4.7) in two ways, we obtain

$$\frac{x^{\cdot}}{\lambda + 1} + Ax = \frac{\Gamma p}{\sqrt{(\Gamma p, p)}}, \qquad \frac{\Gamma^{-1}x^{\cdot}}{\lambda + 1} + \Gamma^{-1}Ax = \frac{p}{\sqrt{(\Gamma p, p)}} \quad (4.8)$$

By pairwise scalar multiplication of the right-hand and left-hand parts of these equations, we obtain

$$\left(\frac{\Gamma^{-1}x^{\cdot}}{\lambda + 1} + \Gamma^{-1}Ax, \quad \frac{x^{\cdot}}{\lambda + 1} + Ax\right) = 1 \quad (4.9)$$

From this quadratic equation we find the root $\lambda + 1$ satisfying the problem and then find the function $F(x, x^{\cdot})$ by Formula (2.12)

for $1 - (\Gamma^{-1}Ax, Ax) \neq 0$

$$F(x, x^{\cdot}) = f_0(x)\,\frac{(\Gamma^{-1}x^{\cdot}, Ax) + \sqrt{(\Gamma^{-1}x^{\cdot}, Ax)^2 + (\Gamma^{-1}x^{\cdot}, x^{\cdot})\,[1 - (\Gamma^{-1}Ax, Ax)]}}{1 - (\Gamma^{-1}Ax, Ax)} \quad (4.10)$$

for $1 - (\Gamma^{-1}Ax, Ax) = 0$

$$F(x, x^{\cdot}) = -\frac{f_0(x)\,(\Gamma^{-1}x^{\cdot}, x^{\cdot})}{2\,(\Gamma^{-1}x^{\cdot}, Ax)} \quad (4.11)$$

Since the function $F(x, x^{\cdot})$ is positive definite for any $x^{\cdot}$ if

$$1 - (\Gamma^{-1}Ax, Ax) > 0,$$

it follows that the region of controllability in any case includes the interior of the ellipsoid whose equation is $(\Gamma^{-1}Ax, Ax) = 1$. The matrix A is here assumed to be nonsingular. We may point out the following geometrical interpretations of the optimum curves of this problem: within the ellipsoid $(\Gamma^{-1}Ax, Ax) = 1$, the optimum trajectories will be the geodesics of Finslerian geometry ([8], chapter 10).

For the problem (1.1) to (4.3), with $f(x) \equiv 0$ in (4.3), we have the function

$$F(x, x^{\cdot}) = \sqrt{(C(x)\,x^{\cdot}, x^{\cdot})}$$

where $C(x)$ is the matrix of the positive definite quadratic form in $x^{\cdot}$. In this case the optimum trajectories are the geodesics of Riemannian geometry.

The author is grateful to E.A.Barbashin for his comments on the results of this article.

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Translated by A.S.